FOSSEE Summer Fellowship Report

On

Mathematical Application Projects using GeoGebra

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Acknowledgment

I would like to express my immense gratitude to the FOSSEE Team at IIT Bombay for allowing me to be a part of this fellowship. It has been a great learning journey. Creating GeoGebra projects to give a visual aid for geometric topics in higher level mathematics is essential to provide better intuitive understanding. This internship has given me the possibility to gain knowledge and experience in triangle centers, circle theorems, packing problems, and other geometric constructions.

I would like to thank Ms. Madhuri Ganapathi, Dr. Snehalatha Kaliappan for their constant guidance and support the entire time. Their valuable suggestions helped me improve my work and learn new concepts. They showed exemplary patience while guiding me through this endeavour. I would also like to express my profound gratitude to Prof. Kannan M Moudgalya for providing me with this opportunity

This fellowship is a major milestone in my professional development. Going ahead, I will try to adapt these skills in my mathematical research. I hope to continue working with FOSSEE and offer more GeoGebra Projects into the Open Access World.

Contents

Chapter 1

Overview

Spoken Tutorial and FOSSEE (Free/Libre and Open Source Software for Education) are projects at Indian Institute of Technology Bombay. They aim to promote the use of open source tools throughout India and reduce the use of proprietary software.

During my three month fellowship, I was tasked to create projects using GeoGebra for Mathematics. To create these projects, I did extensive research, created the constructions on 2D Graphics in GeoGebra, and wrote extensive LaTex documentation that provided a step by step guide for how the project was made.

I provide below an idea of each of my projects, describing in brief, the topic and the construction procedure for each. My contributions include a total of 25 projects spanning mostly dealing with advanced triangle and circle constructions/theorems and packing problems.

Chapter 2

FOSSEE and SOUL

2.1 FOSSEE

FOSSEE (Free/Libre and Open Source Software for Education) project promotes the use of FLOSS tools in academia and research. The FOSSEE project is part of the National Mission on Education through Information and Communication Technology (ICT), Ministry of Education (MoE), Government of India. Science Open source Software for Teaching Learning or SOUL for short is one of the projects which is promoted by FOSSEE.

2.2 SOUL

SOUL (Science OpensoUrce Software for Teaching Learning) is an attempt to put together popular ICT software used as teaching/learning tools by the community of educators and learners in basic concepts and advanced learning of Science subjects. These software can be used as ICT tools for topics in science subjects in classroom teaching and learning.
Learners can learn the software using the resources available for each of these software on SOUL website. They can create a project on a particular topic using the available software.
This website can be used as a platform to showcase the completed projects using the software promoted on this page. These projects will be available as resource material for all the interested educators and learners.

Chapter 3

GeoGebra Constructions

3.1 2D Graphics with GeoGebra

GeoGebra provides a platform for interactive constructions in Mathematics. It gives the opportunity to visually implement abstract concepts into animated projects with no required coding experience. GeoGebra offers a GUI with tabs and buttons of all the most common constructions of lines, angles, circles, and polygons.

The interactive animation can be implemented using sliders, movable points on curves, and checkboxes. The number of advanced concepts that can created with simple lines, curves and polygons is surprisingly large.

3.2 Beginner's Guide to the Interface

I created a Beginner's Guide on all the options in the main tab of the 2D Graphics in GeoGebra so that one can get better acquainted with various possible constructions. A basic understanding of the preliminary tools in Geogebra can be obtained with the help of this guide and side-by-side implementation of each option with the installed or web version of GeoGebra on one's computer.

Chapter 4

Contributions

I worked on the following projects. A short description and a link to the exported webpages of each of the projects is provided below:

4.1 Centers of a Triangle

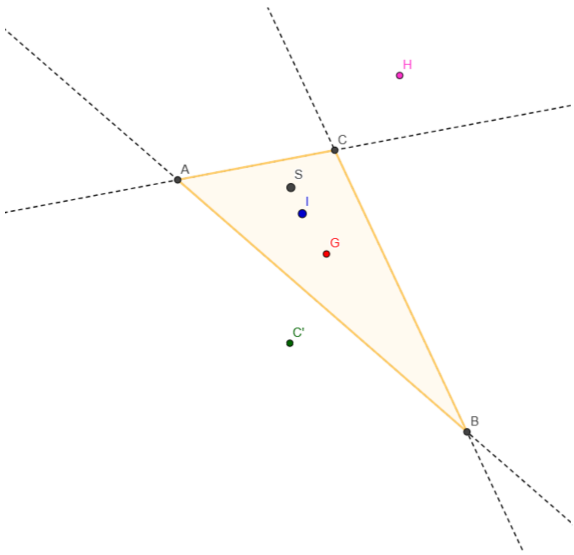


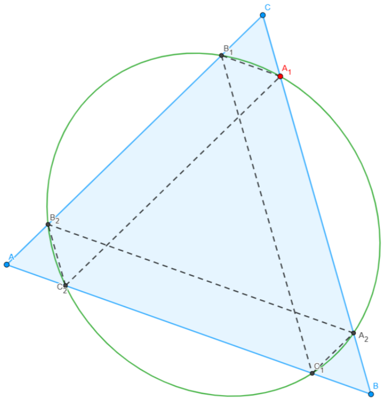
Figure 4.1: Some Triangle Centers

Centers of a Triangle aims to introduce the basic centers related to a triangle. Incenter, circumcenter, centroid, orthocenter, nine point circle, and symmedian point are drawn for a triangle using the 2D construction tools in GeoGebra. The dimensions of the triangle can be adjusted by dragging its vertices. GeoGebra's interactive interface allows us to observe the correlations in positions of the centers of a triangle by varying one or more of the triangle properties.

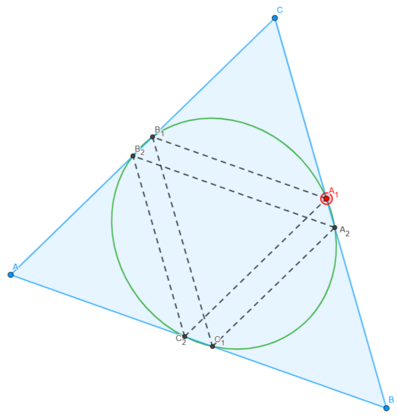
Many centers enumerated in the massive reference of Kimberling Centers can be plotted using only three types of lines: perpendicular bisectors, angular bisectors, and medians. A major research avenue has been to represent all Kimberling centers in terms of the radii of the incircle and circumcircle or coordinates of lower valued

Kimberling centers.

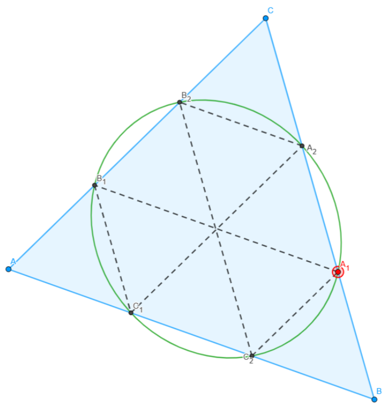
4.2 Thomsen's Figure



(a) Thomsen's Figure with A_1 shown in red.



(b) A_1 approaching the midpoint of side BC.



(c) A_1 at the the trisection points of side BC.

Figure 4.2: Thomsen's Figure in two positions of a triangle ABC starting from a point A_1 movable along side BC

Thomsen's Figure is a geometric construction inside a triangle which proves the following fact: Given a triangle ABC, suppose point A_1 lies on side BC opposite A. A line segment is drawn parallel to AB intersecting AC at point B_1. Now, a segment passing through B_1 and parallel to BC is drawn which intersects AB at C_1. Continuing, we get a segment passing through C_1 parallel to CA intersects BC at A_2, a segment passing through A_2 parallel to AB intersects CA at B_2, and a segment passing through B_2 parallel to BC intersects AB at C_2. Thomsen's theorem shows that the final segment from C_2 parallel to CA intersects the side BC exactly at the original point A_1.

4.3 Tangential Triangle and Lemoine Axis

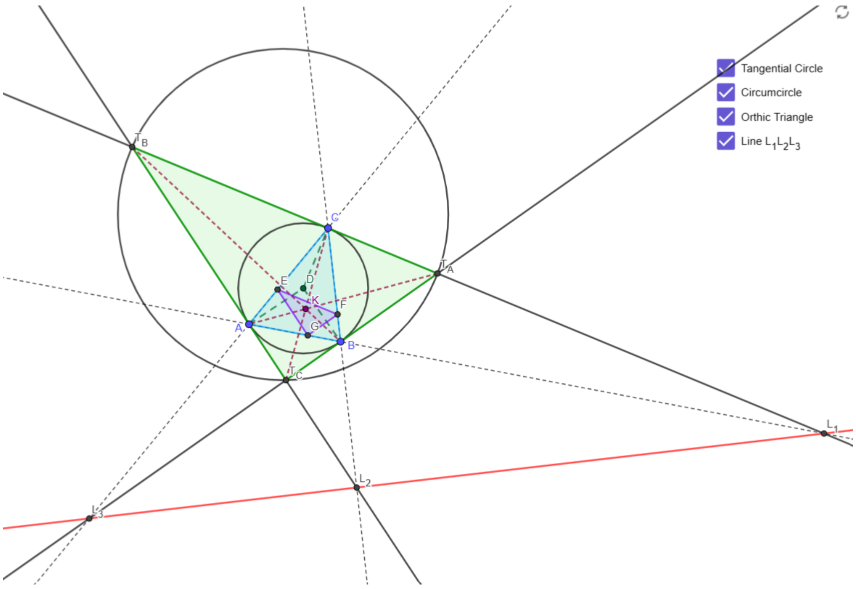


Figure 4.3: Tangential Triangle $T_AT_BT_C$ and Lemoine axis $L_1L_2L_3$ of triangle ABC

Tangential Triangle, Tangential Circle and Lemoine Axis are three related concepts that I chose to group under one project. Given a triangle ABC with circumcircle, i.e. a circle which passes through all three vertices, tangents at A, B, and C are constructed. These tangents intersect two at a time at points T_A, T_B, and T_C where T_I lies on the opposite vertex I for $I = A, B, C$. The triangle $T_AT_BT_C$ is called the tangential triangle and the circumcircle of this circle is called the tangential circle. It is interesting to note that the tangential triangle has sides parallel to the orthic triangle.

The sides of both triangles are extended. The point where the extended sides AB meets T_AT_B is labelled L_1. Similarly the extended sides AB meets T_BT_C at L_2 and AB meets T_AT_C at L_3. The points L_i lie on a straight line. The Lemoine Axis is the line $L_1L_2L_3$.

Let T_AA, T_BB and T_CC intersect at the point K. The tangential triangle and reference triangle are said to be "in perspective" with perspectix as the Lemoine Axis and perspective center as the point K. Two triangles are in perspective if the three pairs of corresponding sides when extended intersect at three points which are collinear $L_1L_2L_3$. The line containing the three points is called the perspectrix. The intersection of the lines through the corresponding vertices is coincident at a point called the perspective center K.

4.4 Simson's Line

Simson's Line Given a triangle ABC and a point P on its circumcircle, perpendiculars are dropped from P to each of the sides of the triangle. The points where these perpendiculars meet the corresponding sides are the three closest points to P on each of the sides of the triangle. These three points are always collinear. The line $P_{AB}P_{BC}P_{CA}$ is called Simson's Line.

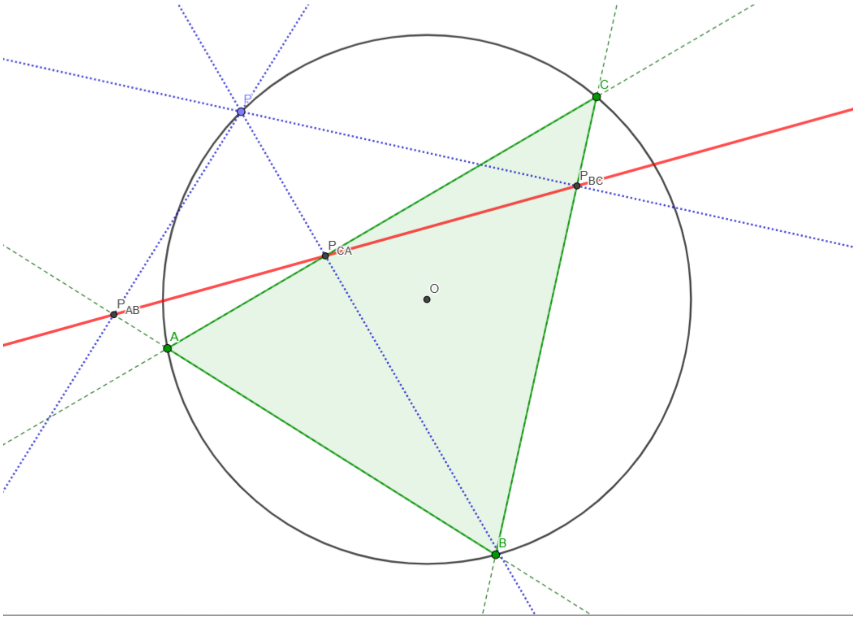


Figure 4.4: The Simson Line given in red for the triangle ABC

4.5 Inner and Outer Soddy Circles

Inner and Outer Soddy Circles are very useful constructions and set a foundation for delving deeper into interesting engineering applications of surprisingly simple geometric constructions.

Given are any three arbitrary non-collinear points A, B, and C. A set of three tangential circles can be drawn using A, B, and C as centres. Furthermore, two circles can be drawn that are tangential to the above three circles simultaneously in the interior and exterior sense. These are known as inner and outer Soddy circles.

These set of five mutually tangential circles with six common tangent points and tangent lines forms an upper limit i.e. no other non-intersecting circles can be drawn tangential to all three circles other than the inner and outer Soddy circles.

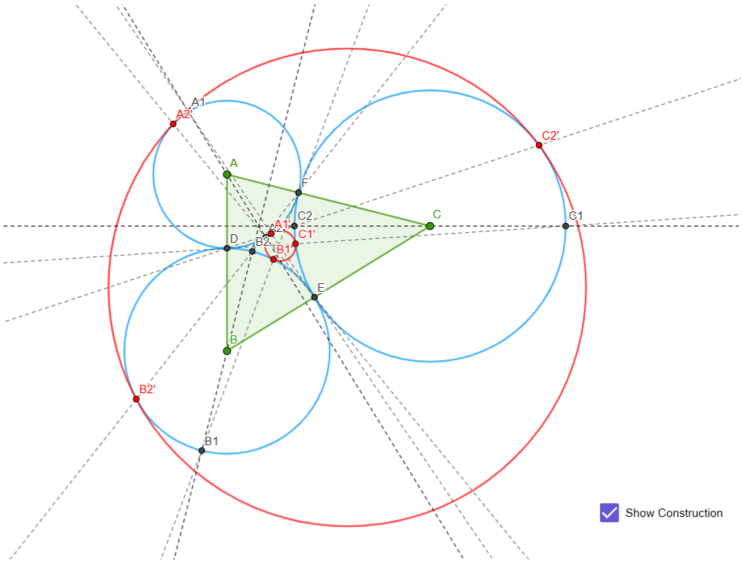


Figure 4.5: Inner and Outer Soddy Circles of a reference triangle ABC

4.6 Apollonian Gasket

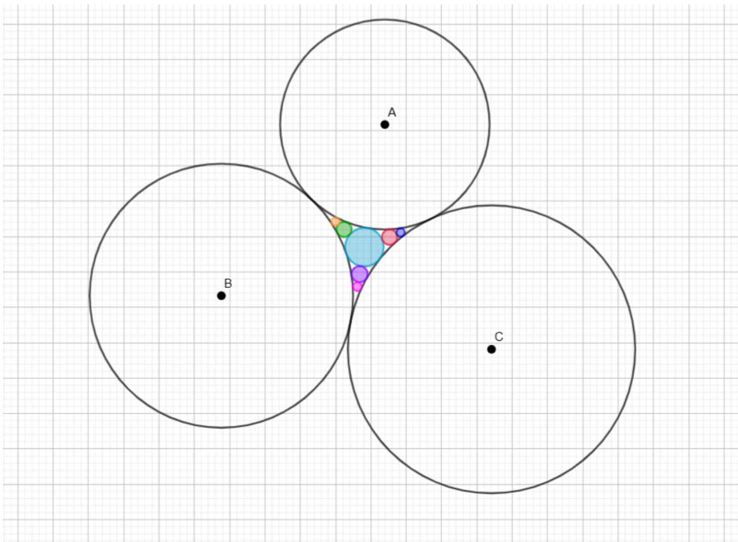


Figure 4.6: Apollonian Gasket

Apollonian Gasket is an application of inner Soddy circles described above. The set up stays the same i.e. three mutually tangential circles with a delta gap in the middle. When Inner Soddy Circles are repeatedly created to fill in the delta gap of the tangential circles, a figure known as the Apollonian Gasket is formed. The construction involves the same steps as mentioned above but continuously redefines the three reference circles to include the Soddy circle recently created. Using the centers of the newly referenced circles, a new reference triangle is defined and is subsequently used for the construction of more inner Soddy Circles.

4.7 Schiffler Point

Schiffler Point is the point of intersection of the Euler lines of the triangles $\triangle ABC$, $\triangle XAB$, $\triangle XBC$, $\triangle XCA$, where X is the incenter of the triangle ABC. The Euler line of a triangle is the line containing the orthocenter, centroid, and circumcenter. Any two of the points may be used to draw the Euler line for each of the triangles specified.

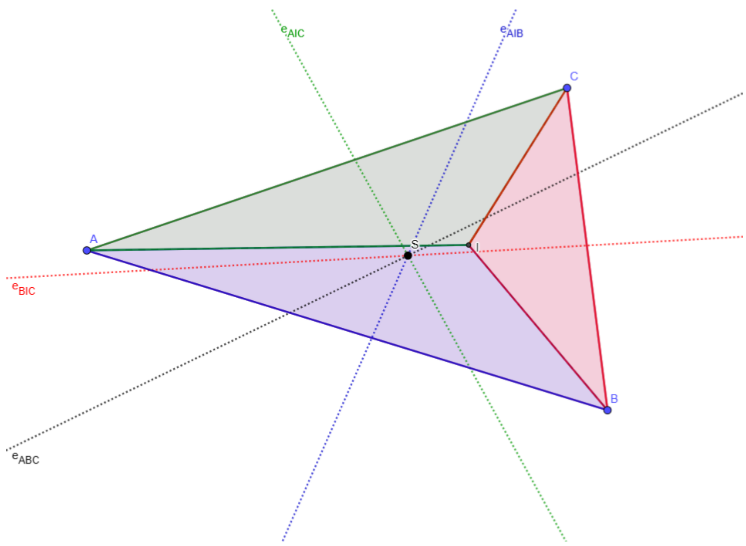
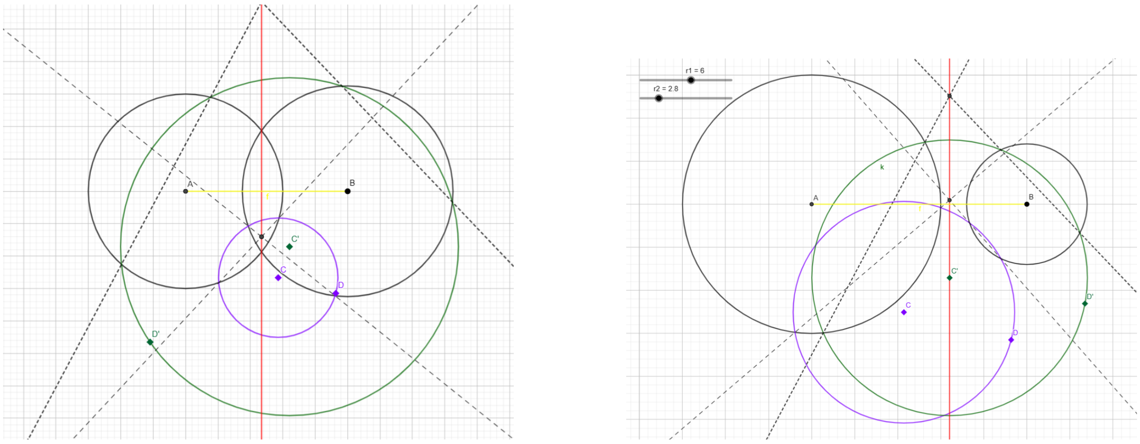


Figure 4.7: Schiffler Point S of a triangle ABC

4.8 Radical Line

Radical Line or radical axis is the locus of points with equal circle power with respect to two different circles centered at A and B. If the circles are tangential, the common tangent is the radical axis. If the circles intersect in two points, then the radical line passes through the points of intersection. If not, two circles are constructed with centers C and C' and passing through points D and D'. These four points are movable and the two circles with centers C and C' should be adjusted such that each circle intersects the circles centered at A and B at two points each. Now, lines are drawn through each pair of points of intersection of each circle. The line connecting their two points of intersection is then the radical line.



(a) Radical line of two intersecting circles centered at A and B

(b) Radical line of two non intersecting circles centered at A and B

Figure 4.8: Radical Line for circles centered at A and B

4.9 Radical Center

Radical Center is the concurrent point of three radical lines with respect to three circles taken two at a time. The radical axis is drawn in the manner specified previously.

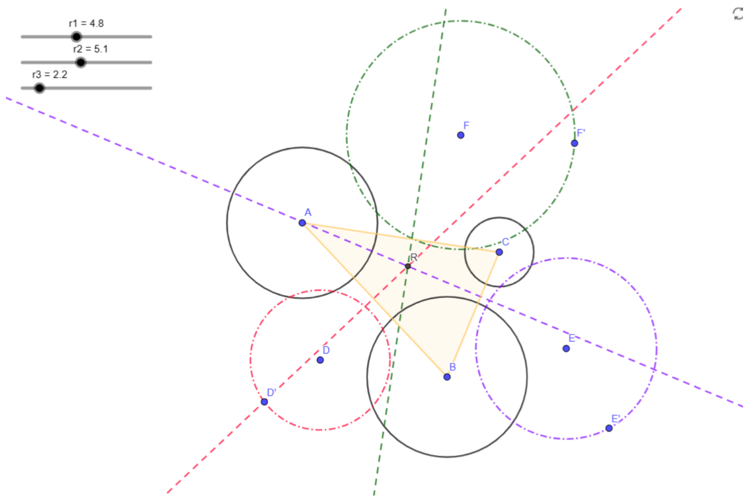


Figure 4.9: Radical Center R of triangles centered at A, B, and C

4.10 Prasolov Point

Prasolov Point is the perspective center of the reference triangle ABC and the reflection of the orthic triangle over the center of the nine- point circle denoted as $A'B'C'$. The orthic triangle is the triangle $H_AH_BH_C$ where H_A, H_B, and H_C are the foot of the perpendiculars from each vertex of triangle ABC. The nine-point circle is the circle which passes through H_A, H_B, and H_C. Its center is the nine-point center. When the orthic circle is reflected over the nine-point center, we get the triangle $A'B'C'$. Lines AA', BB' , and CC' are drawn which are concurrent at the Prasolov Point P.

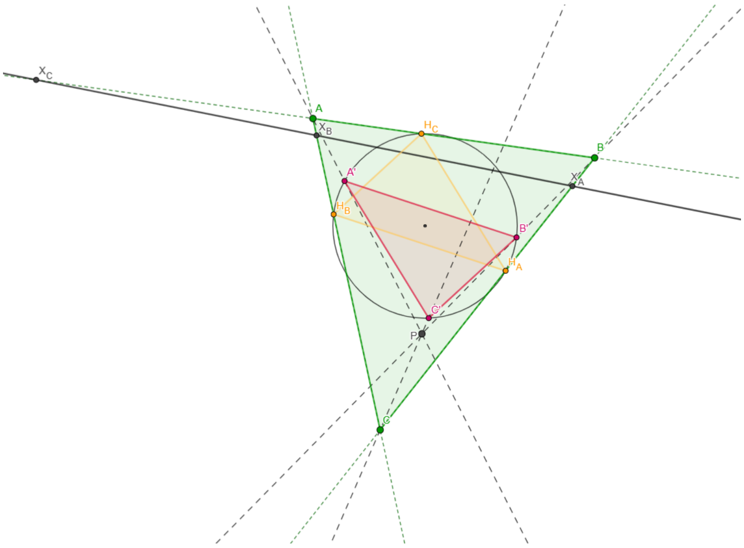


Figure 4.10: Prasolov Point P of the triangle ABC with Orthic Triangle $H_AH_BH_C$

4.11 Adam's Circle

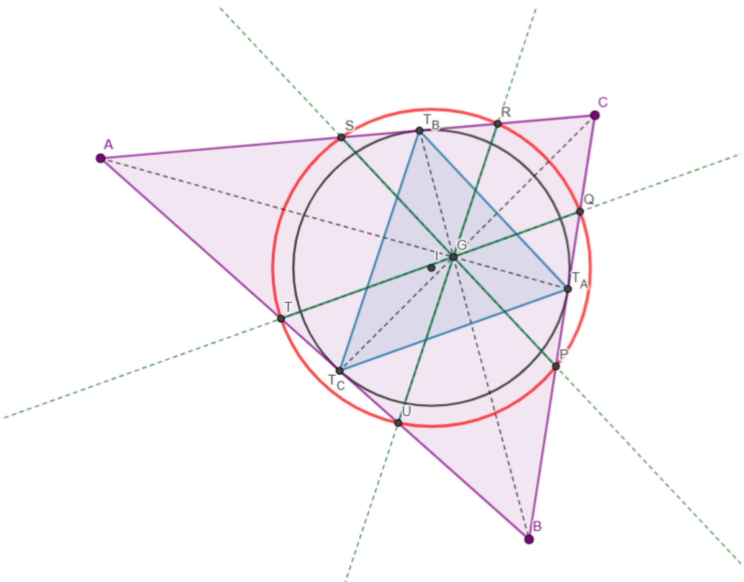


Figure 4.11: Adam Circle for a triangle ABC

Adam's Circle is a circle made from a specified set of six concyclic points. Given triangle ABC, draw the incircle of the triangle. Now the points where the incircle is touches the sides of the triangle ABC forms a triangle known as the contact triangle

$T_A T_B T_C$. The perspector of the triangles ABC and $T_A T_B T_C$ is a point known as the Gergonne point G. Now lines passing through G and parallel to each of the sides of the contact triangle are drawn. These lines intersect the reference triangle ABC at six points which are concyclic and form the Adam Circle.

4.12 Antiparallels

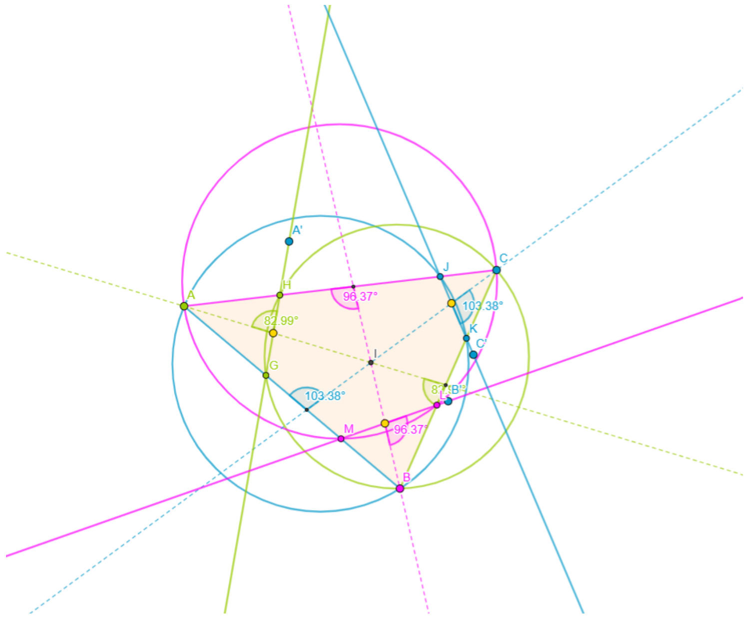


Figure 4.12: Antiparallels JK, GH, and LM to the sides AB, BC, and CA respectively

Antiparallels are any pairs of lines PQ and RS such that they make the same angle in the opposite senses with the bisector of a particular angle of the triangle. It is a relatively simple reasoning to show that if PQ and RS are antiparallel, then the points P, Q, R, and S are concyclic. Antiparallel lines appear very commonly in geometric constructions. For exampl,e the line joining the feet to two altitudes of a triangle is antiparallel to the third side. The tangent to a triangle's circumcircle at a vertex is antiparallel to the opposite side. Also, the radius of the circumcircle at a vertex is perpendicular to all lines antiparallel to the opposite sides.

4.13 Circle Power

Circle Power of a point A with respect to a circle of radius r centered at O is a numerical value that takes on different values depending on their relative position. If A lies on the circle, its power is 0. If A lies outside the circle, the tangent to the circle through A is drawn. The point where the tangent touches the circle is marked as T. The circle power is given by $|AT|^2$. Equivalently, any line is drawn through A intersecting the circle at two points, say P and Q. Then the circle power is $|AP| \times |AQ|$. If A lies inside the circle, the power of A is half the length of a chord of the circle passing through A and perpendicular to OA.

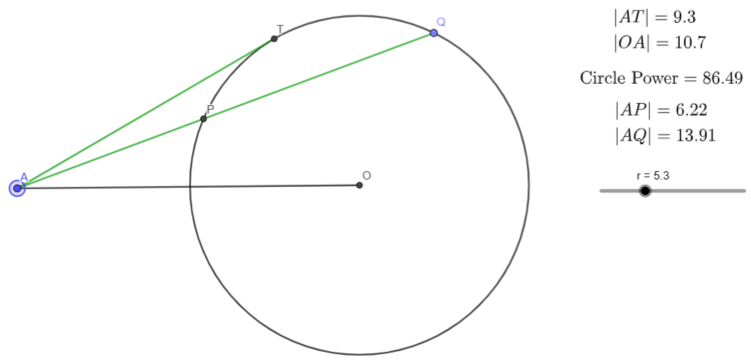


Figure 4.13: Circle Power of a Point A with respect to a circle centered at O with radius r

4.14 Conway Circle

Conway Circle is a circle defined by six points on the extended sides of a reference triangle ABC. let the lengths of sides BC, CA, and AB be a, b, and c respectively. Define A_b to be a point on the extension of side CA beyond A such that $AA_b = a$. Define C_b as the point on the extension of side CA beyond C such that $CC_b = c$. Points B_c and B_a lie near B on extended sides AB and BC respectively with $BB_c = b = BB_a$. Similarly, C_a and C_b lie near C on the extended sides BC and CA respectively with lengths $CC_a = c = CC_b$. The points A_b, A_c, B_c, B_a, C_a, and C_b are concyclic and the resulting circle is known as Conway circle of $\triangle ABC$.

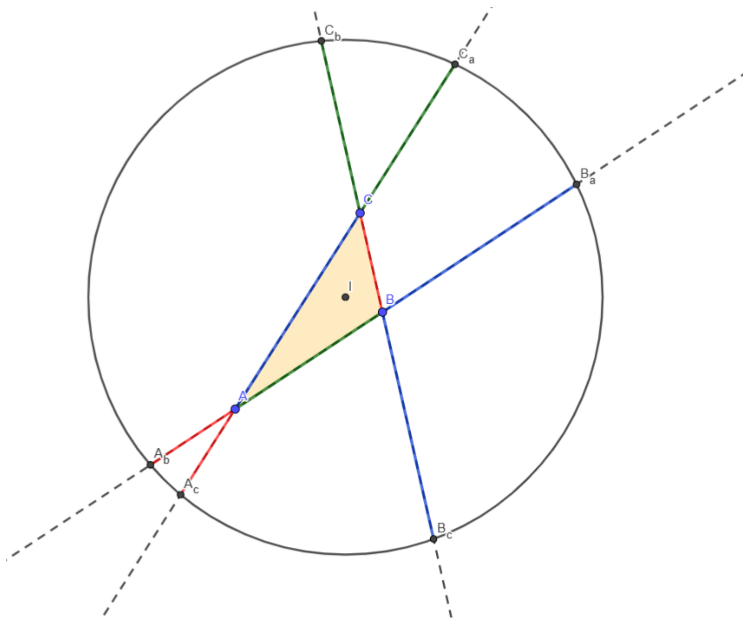


Figure 4.14: Conway Circle of a triangle ABC

4.15 Feuerbach Point and Antipode

Feuerbach Point is the point where the incircle and nine-point circle are internally tangent. The antipode of a point with respect to a circle is the point diametrically opposite. The construction is easily done directly from definitions.

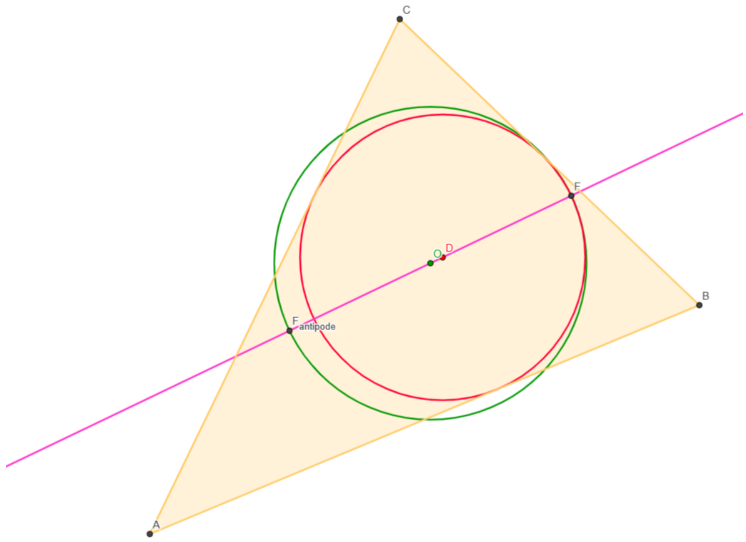


Figure 4.15: Feuerbach Point F and its antipode for a triangle ABC

4.16 Fonetene's Theorems

This project includes explanatory construction of Fontene's first, second, and third theorems to give an idea about the interconnection of various constructions surrounding the pedal circle. Specifying points of concurrency, tangency and intersection, these theorems help to understand the theory behind pedal circles given any points inside a reference triangle.

The Fontené Theorems are a set of three theorems relating to the pedal circle of a triangle. Suppose we are given any point P and a reference triangle. Three perpendiculars are dropped from P to each of the sides of the reference triangle. The points where these perpendiculars intersect the corresponding sides give the vertices of the pedal triangle. The circle drawn through these three vertices is the pedal circle. The three points where the medians intersect the corresponding sides of the triangle are the vertices of the so called medial triangle and its circumcircle is the medial circle.

4.16.1 Fontené's First Theorem

Fontené's first theorem proposes an interesting insight using only the simple constructions mentioned above. First, it is noted that the intersection of corresponding

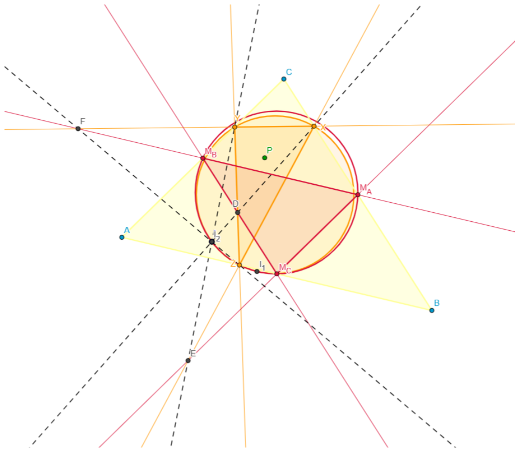


Figure 4.16: Construction depicting Fontené's first theorem

extended sides of the pedal triangle XYZ and medial triangle $M_A M_B M_C$ gives a set of three points say, D, E, F. The lines XD, YE, ZF form a set of concurrent lines whose point of concurrency is precisely the intersection/tangency of the medial and pedal circles.

4.16.2 Fontené's Second Theorem

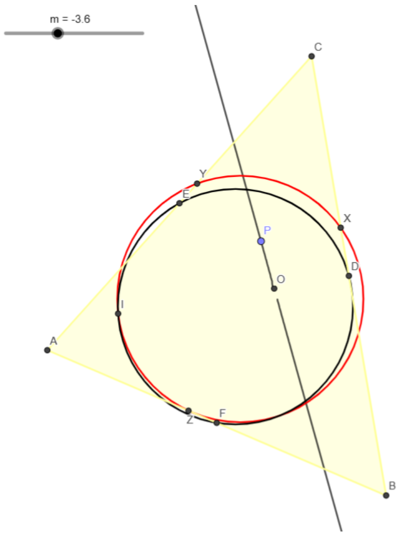


Figure 4.17: Construction depicting Fontené's second theorem

The nine point circle is a circle passing through the perpendicular feet and the median feet of the reference triangle. Fontené's Second Theorem says that if a point P moves on a fixed line through the circumcenter of the reference triangle, its pedal circle passes through a fixed point on the nine-point circle.

4.16.3 Fontené’s Third Theorem

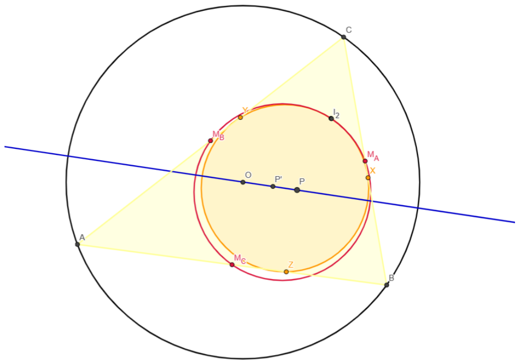


Figure 4.18: Construction depicting Fontené’s third theorem

The cevians of a triangle with respect to a point is a set of three concurrent lines joining each vertex to that point. The isogonal conjugate of a point is obtained by reflecting these cevians along the corresponding angular bisectors. The Fontené third theorem states that given a point P and a reference triangle, its pedal circle and the nine-point circle are tangent iff P and its isogonal conjugate P’ lie on a line through the circumcenter.

4.17 Packing Problems

Packing problems include the entire branch of mathematics concerned with optimizing the area a fixed number of objects with a fixed length can occupy. The goal is to pack what is known as a ”container” as densely as possible with shapes of unit side length or unit radius. Many of these problems can be related to real-life packaging, cutting, storage and transportation issues. I looked into a few of the basic packing problems and created GeoGebra projects for them.

4.17.1 Unit Squares in an Equilateral Triangle

Here, the goal is to find the minimum side length of an equilateral triangle in which n unit equilateral triangles can be packed (i.e. packing of n unit equilateral triangles into an equilateral triangle as densely as possible without overlap). In many cases the values of angles are degrees with large decimal suffixes and have been obtained by rigourous computer led calculations. For our purposes, approximate constructions are used instead which give correct values to a sufficient amount of significant figures such that the distinction is not perceptible in the GeoGebra image.

4.17.2 Unit Equilateral triangles in an Equilateral Triangle

Here, the goal is to find the minimum side length of an equilateral triangle in which n unit equilateral triangles can be packed (i.e. packing of n unit equilateral triangles into an equilateral triangle as densely as possible without overlap).

4.17.3 Unit Circles in a Square

Here, the goal is to find the minimum side length of a square in which n unit circles can be packed (i.e. packing of n unit circles into a square as densely as possible without overlap).

4.18 Covering of Circles with Unit Circles

Each packing problem has a dual covering problem, which asks how many of the same objects are required to completely cover every region of the "container", where objects are allowed to overlap. The GeoGebra constructions provided have been made to an accuracy of two decimal places. Further precision can be obtained with deeper mathematical background or computer obtained construction values.

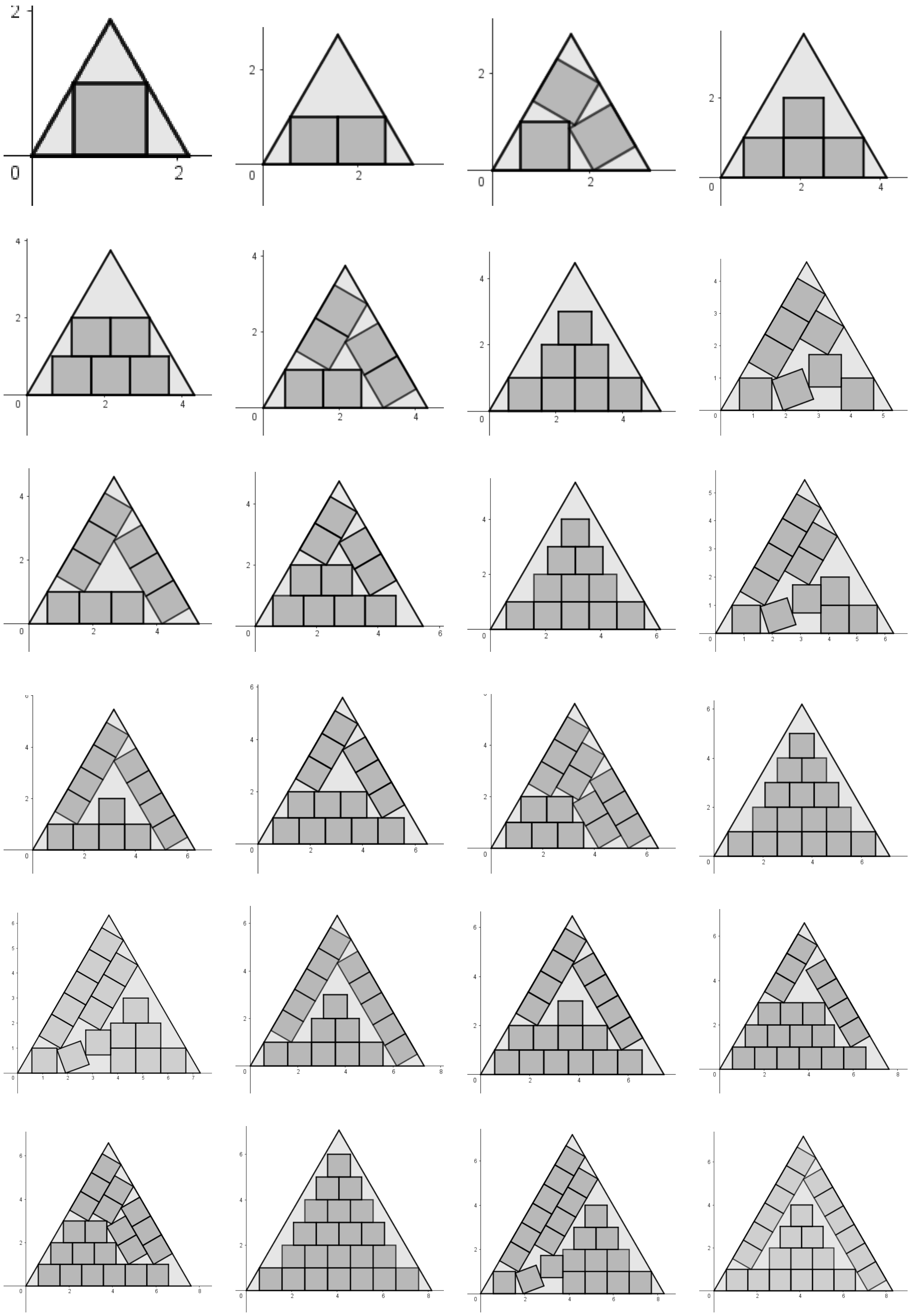
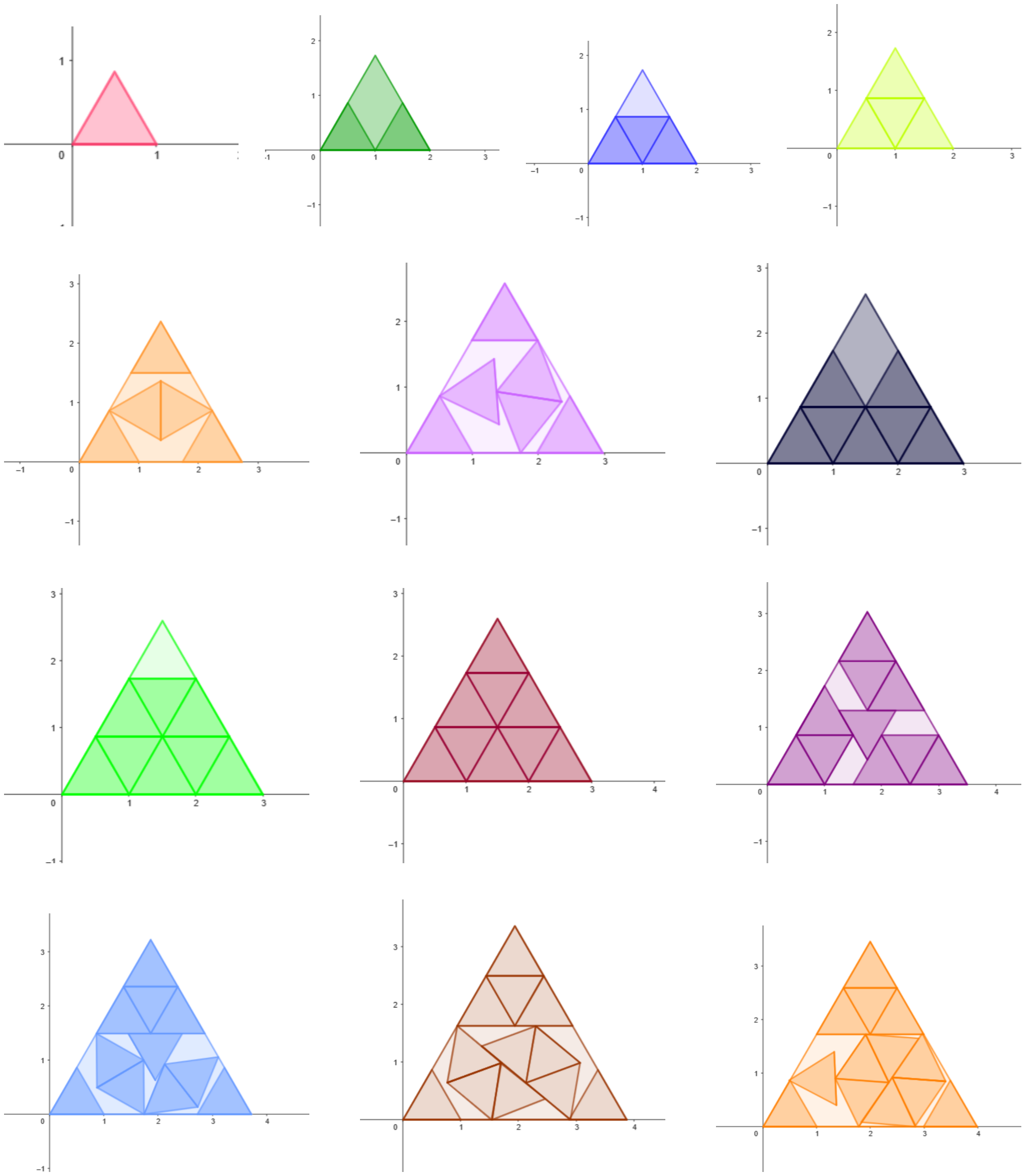


Figure 4.19: 1 - 24 Unit Squares in an Equilateral Triangle



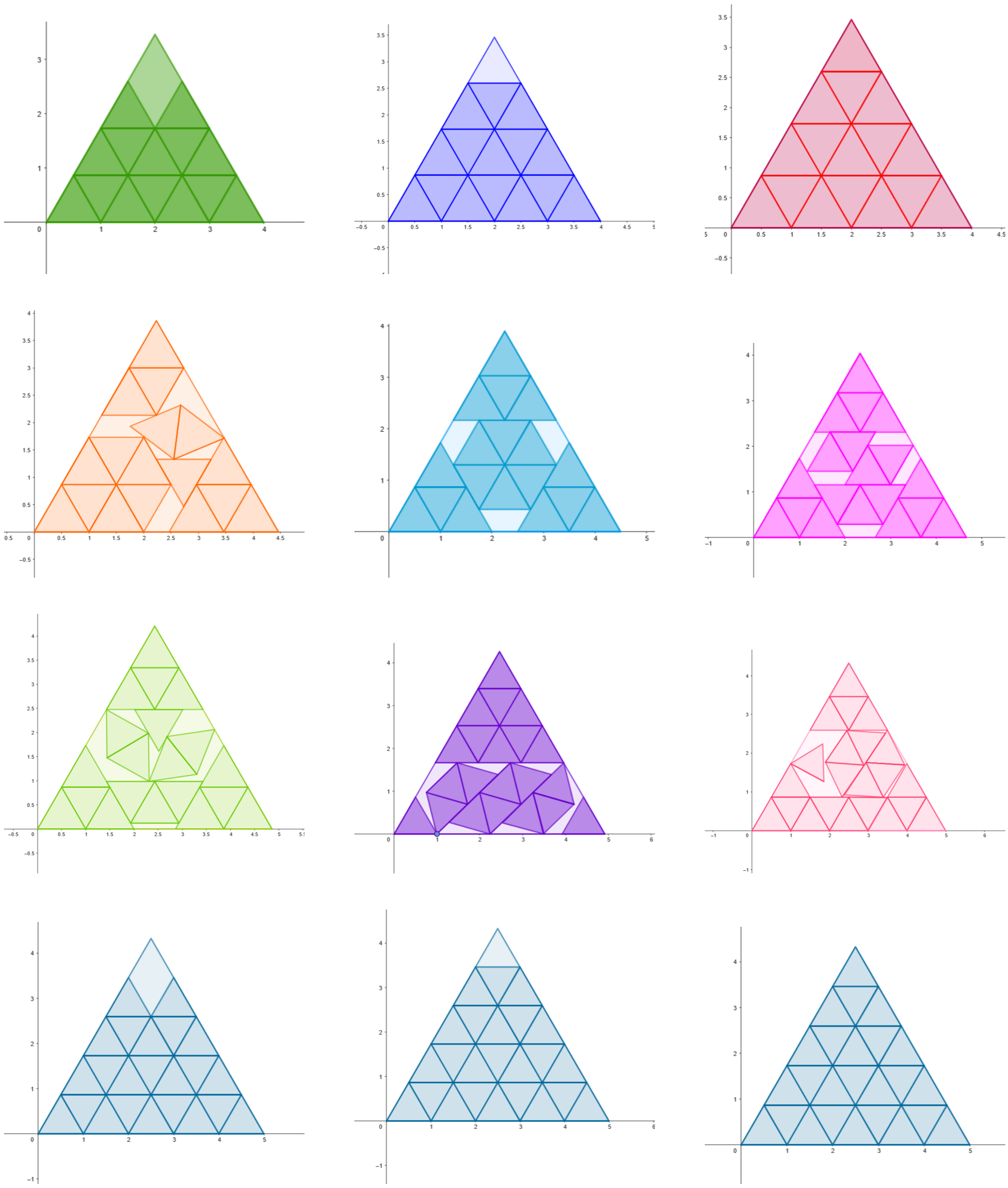


Figure 4.20: 1 - 25 Unit Equilateral Triangles in an Equilateral Triangle

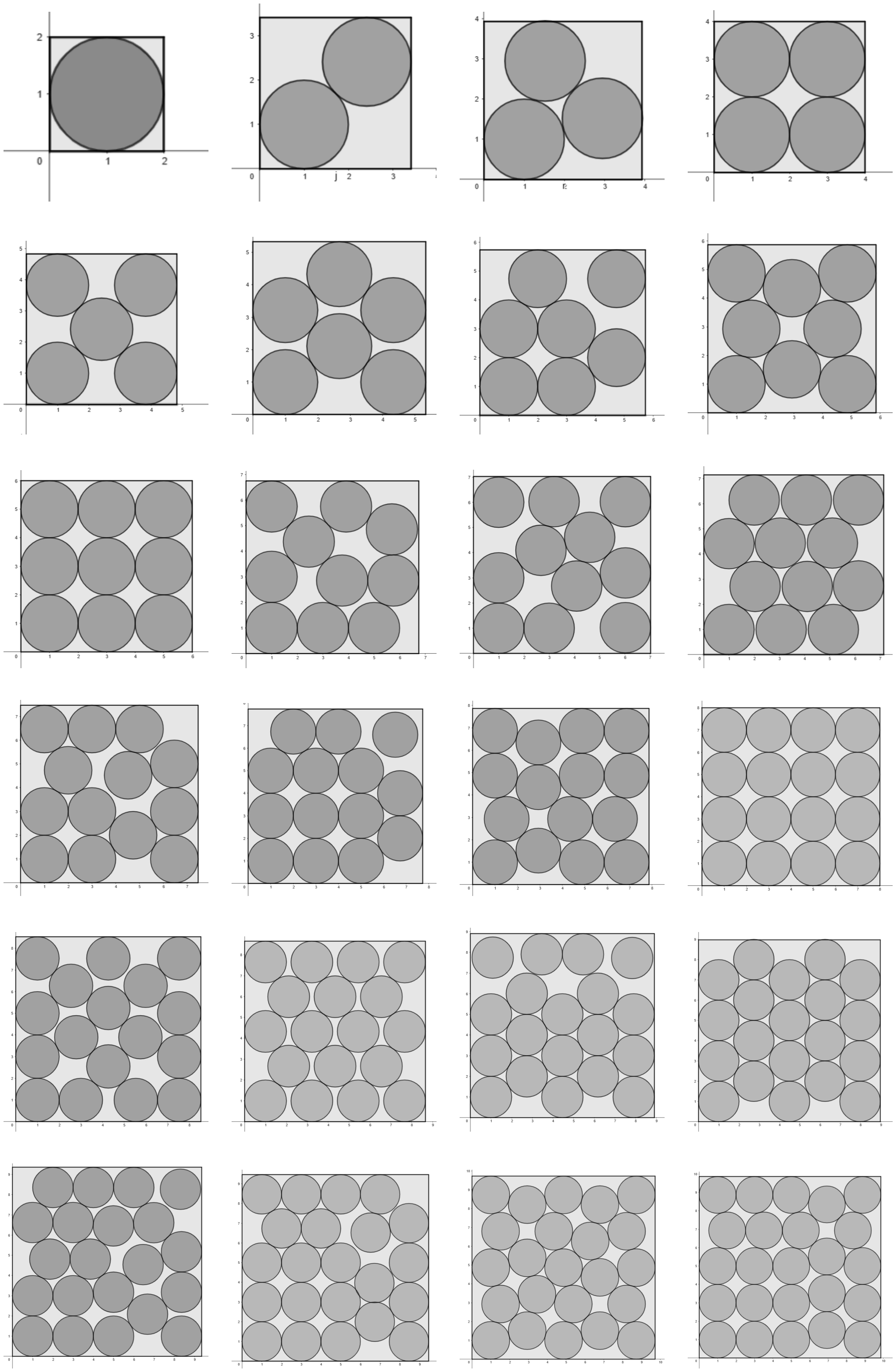
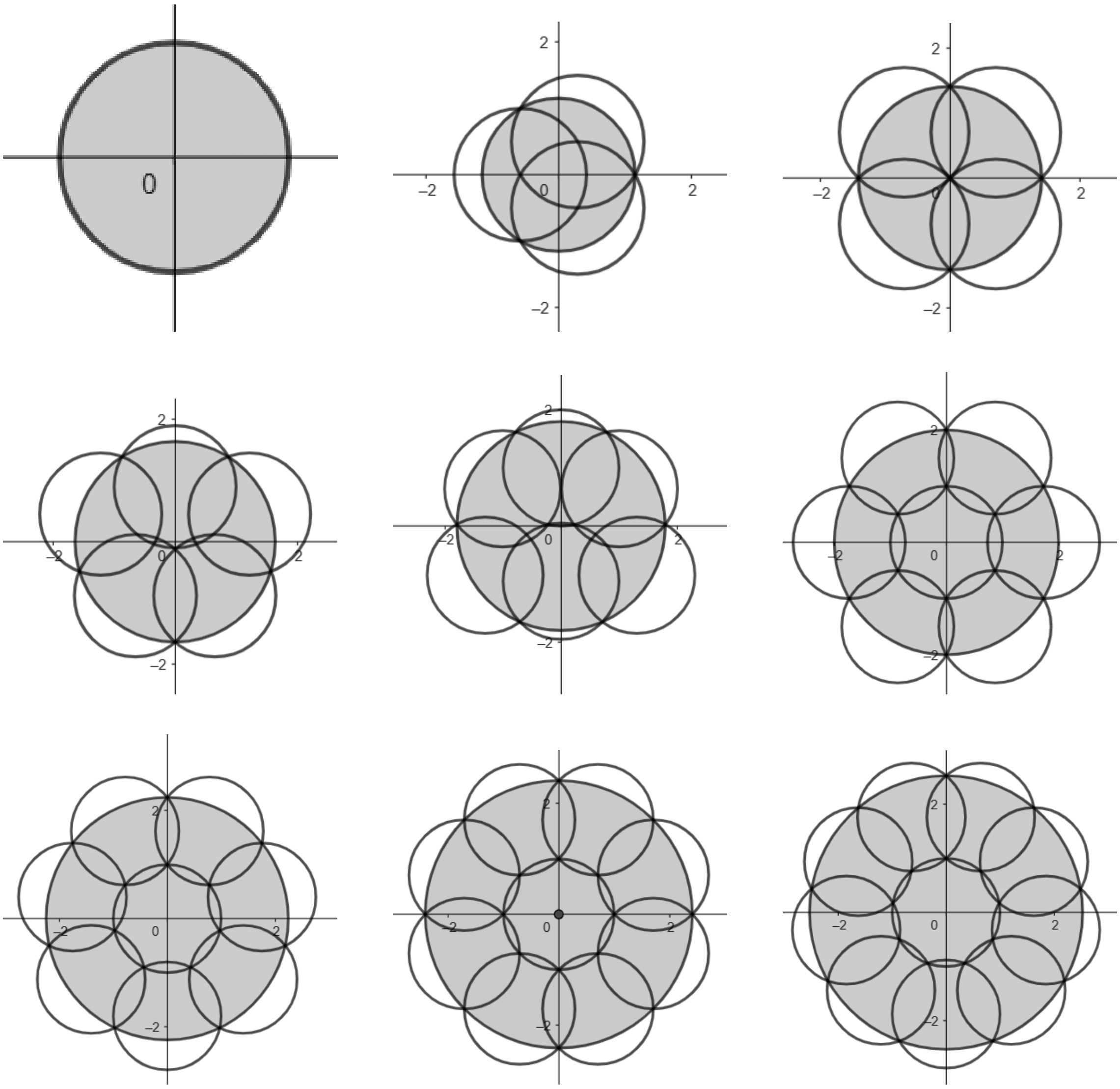


Figure 4.21: 1 - 24 Unit Circles in a Square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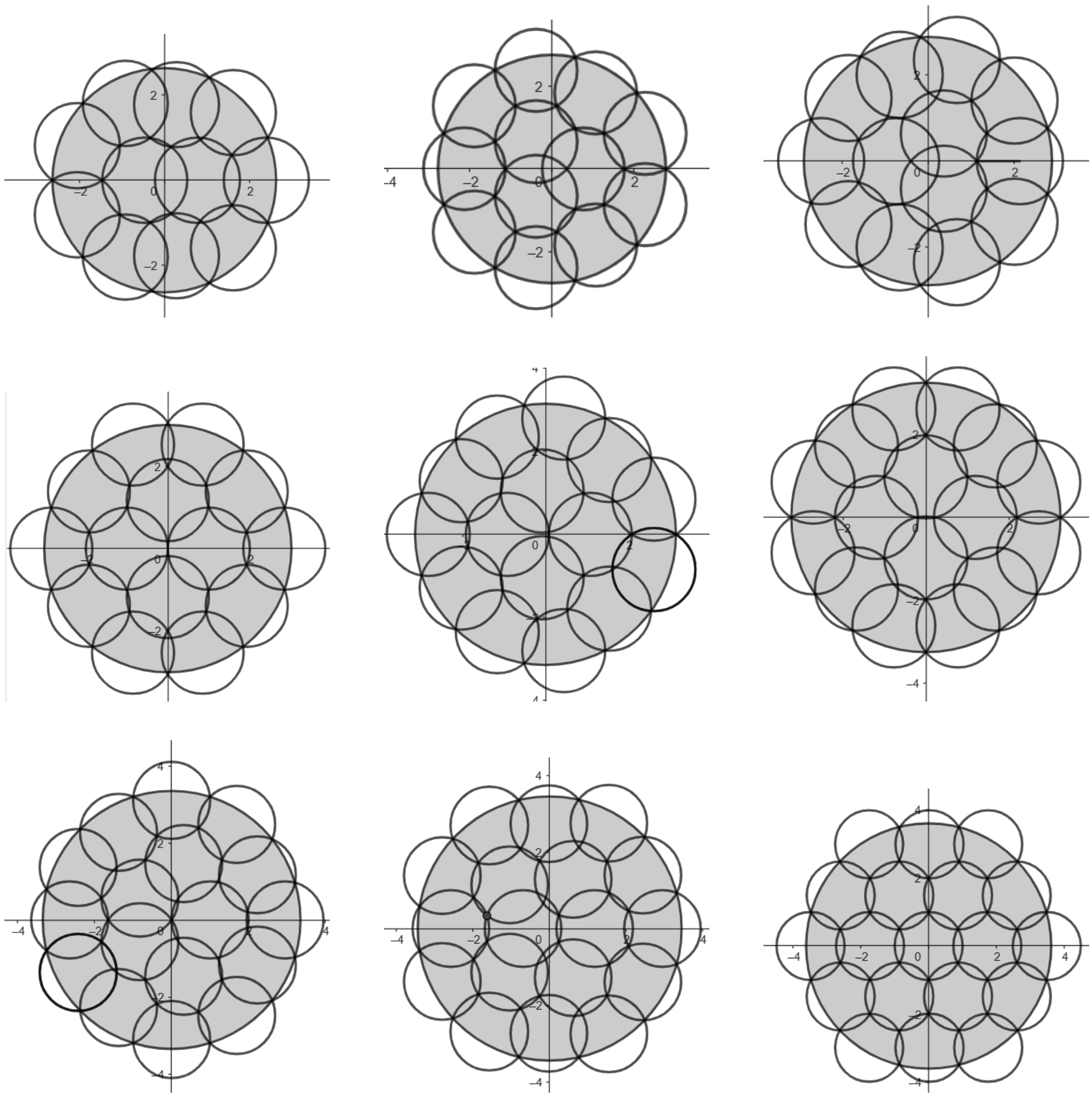

Figure 4.22: 1 - 19 Unit Circle Coverings of Circles

Chapter 5

Professional outcomes

Professional skills developed during this internship are:

- Workplace communication skills
- Time management
- Creating work reports using LaTeX

Chapter 6

Challenges

Challenges that I faced during the fellowship:

- Figuring out which mathematical concepts can be best explained using GeoGebra applications.
- Understanding intuitive reasoning behind constructions.
- Familiarizing myself with the GeoGebra interface.
- Implementing worded instructions using drawing tools in GeoGebra keeping in mind orientation (of vertices of a triangle), variable names (of centers, midpoints, reflections, etc.), and line characteristics.
- Adjusting design and style settings so as to enhance visibility and readability of the final constructions without hiding intermediate steps.
- Creating animated constructions that can incorporate multiple orientations/ positions into one project.
- Creating solutions to packing problems for various unit regular polygons on a case by case basis.
- Writing a coherent and understandable documentation for complex constructions
- Using correct and uniform terminology/syntax throughout the documentation.

Chapter 7

Conclusion

In conclusion, being a part of the FOSSEE Summer Fellowship 2023 was a remarkable experience. It helped me gain new knowledge in 2D GeoGebra construction. I got a chance to see the high prospects of using open source software when teaching mathematics both at the elementary and college level.

I became acquainted with various centers of triangles and foundational theorems related to pedal circles, tangential circles and perspective triangles. I also learned about packing problems and came to appreciate the immense amount of programming based research that went into proving maximal area packing/covering. using GeoGebra helped showcase the practical aspects of these constructions and was a great way to get a better understanding of the concepts without the need of rote learning.

I would like to express my gratitude to my mentors and the extended FOSSEE team who helped make this fellowship a great experience.

Chapter 8

Useful Links

8.1 Documentation of Constructions

- Thomsen's Figure
- Tangential Triangle, Tangential Circle and Lemoine Axis
- Simson's Line
- Inner and Outer Soddy Circles
- Schiffler Point
- Radical Line
- Radical Center
- Prasolov Point
- Adam's Circle
- Circle Power
- Conway Circle
- Feuerbach Point and Antipode
- Fonetene's Three Theorems

8.2 Reference

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